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# Topological Structure of Entropy of 4-Dimensional Axisymmetric Black Holes

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**Abstract** Using the relationship between the entropy and the Euler characteristic, an entropy density is introduced to describe the inner topological structure of the entropy of 4-dimensional axisymmetric black holes. It is pointed out that the density of entropy is determined by the singularities of the timelike Killing vector field of spacetime, and these singularities carry the topological numbers, Hopf indices and Brouwer degrees, which are topological invariants. At last, Kerr–Newman black hole as an example of axisymmetric black holes is given. What's more, the entropy and the latent heat of the topological phase transition of the black hole mentioned above are calculated and the latent heat just lies in the range of the energy of gamma ray bursts.

**Keywords** Euler characteristic · Entropy · Kerr–Newman black hole · Killing vector field

## 1 Introduction

At the beginning of the 1970s, an impressive series of works on black hole thermodynamics was done [1–3]. It was found that the Bekenstein–Hawking entropy $S$ of black holes had a relationship with the area $A$ of event horizon

$$S = \frac{A}{4}. \tag{1}$$

After Hawking's discovery of black hole radiation [4, 5], a framework about the entropy of black holes was formed, which resulted from the application of quantum field theory to such a peculiar spacetime.

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However, in decades years, it is pointed out that the (1) is not suitable for extremal black holes [5, 6]. Because the area of the horizon of 4-dimensional extremal black hole is nonvanishing, but the entropy is zero. It indicates that the topology of spacetime plays an essential role in the explanation of entropy of black holes. The extremal black holes have a nontrivial topological structure, which leads to the vanishing of Euler characteristic. But these results meet some challenges. For example, it has been supported by string theorists, who have derived at the result by counting string states [7], that the entropy of extremal black holes is still proportional to the area of horizon. In particular, some of these black holes have their microscopic degrees of freedom accounted for within string theory. Hence, there is apparently a distinction between thermodynamic entropy and entropy results from a large number of degenerate vacua. Meanwhile the string theorists' results were interpreted by summing over the topology [8]. They seem to suggest that the extremal black holes should be considered as a rather different object from the nonextremal ones.

In 1997, by calculating almost all known gravitational instantons, Liberati and Pollifrone presented a new formulation of the Bekenstein–Hawking law [9], which gave the relationship between the entropy $S$ and the Euler characteristic $\chi$ as

$$S = \frac{A}{8}\chi. \tag{2}$$

This new formulation was shown to be valid for almost all known extremal and nonextremal black holes. In this paper, we will investigate the topological structure of the entropy of 4-dimensional axisymmetric black holes by using this new formulation. Because their thermodynamical behavior is very different from the spherically symmetric black holes, which has its origin in their much more complicated causal structure. Hence, their study is of great interest in understanding physical properties of astrophysical objects, as well as in checking any conjecture about thermodynamical properties of black holes. In differential geometry, there are two ways to obtain the Euler characteristic of a manifold. One is from the Gauss–Bonnet–Chern (GBC) theorem to calculate the volume integral of curvature tensor. It gives the global structure of the whole manifold. The other is in terms of the Hopf index theorem to sum the indices of singularities of a smooth unit tangent vector field defined over the manifold. In our opinion the Hopf theorem shows the inner structure of the topology of manifold, because for the same Euler characteristic, there can be different combinations of both the singularities and their indices. So, we will reach our aim through these singularities and indices. For the case considered here, the smooth unit tangent vector field coincides with the timelike Killing vector field of spacetime [9].

This paper is organized as follows. In Sect. 2, the Euler characteristic and the entropy density of 4-dimensional axisymmetric black holes are introduced. In Sect. 3, the intrinsic topological structure of entropy is studied by using the Killing vector field. Then, Kerr–Newman black hole as an example of 4-dimensional axisymmetric black holes is given, its Euler characteristic, the entropy and the latent heat are calculated in Sect. 4. The last Sect. 5 is a conclusion of this paper.

## 2 The Euler Characteristic and the Entropy Density of 4-Dimensional Axisymmetric Black Holes

It is well known that the metrics of 4-dimensional axisymmetric black holes in a normal coordinate are of the form

$$ds^2 = e^{2\overline{\nu}}dt^2 - e^{2\Psi}(d\varphi - \omega dt)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2, \tag{3}$$

where $\overline{\nu}$, $\Psi$, $\mu_2$ and $\mu_3$ are the functions of the coordinates.

In order to study the intrinsic topological property of entropy, let us consider the Euler characteristic in detail. For a closed $N$(even)-dimensional Riemannian manifold $M^N$, the Euler characteristic $\chi(M^N)$ can be expressed as the volume integral of the Gauss–Bonnet–Chern differential $N$-form $\Lambda$

$$\chi(M^N) = \int_{M^N} \Lambda, \tag{4}$$

$$\Lambda = \frac{(-1)^{N/2}}{2^N \pi^{N/2}(N/2)!} \epsilon_{a_1 a_2 \cdots a_{N-1} a_N} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{N-1} a_N}, \tag{5}$$

where $R^{ab}$ is the curvature 2-form of $M^N$. Based on an instructive idea of working on the sphere bundle $S(M^N)$, Chern [10, 11] has shown that the GBC $N$-form $\Lambda$ on $M^N$ can be pulled back to $S(M^N)$ as the exterior derivative of a differential $(N-1)$-form $\Omega$:

$$\pi^* \Lambda = d\Omega, \tag{6}$$

where $\pi^*$ denotes the pullback of the projection $\pi: S(M^N) \longrightarrow M^N$. Then, using a recursion method and a section $n$: $M^N \longrightarrow S(M^N)$, which is a unit tangent vector field over $M^N$ satisfying

$$n^a(x) n^a(x) = 1 \quad (a = 1, 2, \ldots, N), \tag{7}$$

Chern [12] proved that the $(N-1)$-form $\Omega$ on $S(M^N)$ can be read as

$$\Omega = \frac{1}{(2\pi)^{N/2}} \sum_{k=0}^{N/2-1} (-1)^k \frac{2^{-k}(N-2k-2)!}{(N-2k-1)!k!} \Theta_k, \tag{8}$$

which is called the Chern form with

$$\begin{aligned} \Theta_k = {} & \epsilon_{a_1 a_2 \cdots a_{N-2k} a_{N-2k+1} a_{N-2k+2} \cdots a_{N-1} a_N} n^{a_1} Dn^{a_2} \Lambda \cdots \Lambda Dn^{a_{N-2k}} \\ & \times \Lambda R^{a_{N-2k+1} a_{N-2k+2}} \Lambda \cdots \Lambda R^{a_{N-1} a_N}, \end{aligned} \tag{9}$$

where $Dn^a$ is the covariant derivative 1-form of $n^a(x)$. It is noted that $\pi^*$ maps the cohomology of $M^N$ into that of $S(M^N)$, while $n^*$ performs the inverse operation. Thus $n^*\pi^*$ amounts to the identity and the Euler characteristic $\chi(M^N)$ in (4) can be rewritten as

$$\chi(M^N) = \int_{M_N} \Lambda = \int_{M^N} n^* \pi^* \Lambda = \int_{M^N} n^* d\Omega. \tag{10}$$

In view of decomposition of gauge potential, Duan et al. show that the $(N-1)$-form $\Omega$ can be formulated in terms of the unit tangent vector field $n^a(x)$ cleanly as [13, 14]

$$\Omega = \frac{1}{(N-1)! A(S^{N-1})} \epsilon_{a_1 a_2 \cdots a_N} n^{a_1} dn^{a_2} \Lambda \cdots \Lambda dn^{a_N}, \tag{11}$$

where $A(S^{N-1})$ is the area of $S^{N-1}$

$$A(S^{N-1}) = \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{12}$$

Then the Euler characteristic $\chi(M^N)$ is

$$\chi(M^N) = \frac{1}{(N-1)!A(S^{N-1})} \int_{M^N} \epsilon^{\mu_1\cdots\mu_N} \epsilon_{a_1\cdots a_N} \partial_{\mu_1} n^{a_1} \cdots \partial_{\mu_N} n^{a_N} d^N x. \tag{13}$$

For the 4-dimensional axisymmetric black holes, (13) becomes

$$\chi = \frac{1}{12\pi^2} \int \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c \partial_\rho n^d d^4 x. \tag{14}$$

Hence, according to (2), the entropy of axisymmetric black hole equals

$$S = \int \rho d^4 x, \tag{15}$$

in which the entropy density $\rho$ is introduced by

$$\rho = \frac{A}{8} \frac{1}{12\pi^2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c \partial_\rho n^d, \tag{16}$$

where $A$ is the area of event horizon. In the following, we will consider the intrinsic topological structure of the entropy through the entropy density and the so-called $\phi$-mapping method.

## 3 The Intrinsic Topological Structure of Entropy of 4-Dimensional Axisymmetric Black Holes

Since $n^a(x)$ is a unit tangent vector field, it can, in general, be further expressed as

$$n^a(x) = \frac{\phi^a(x)}{\|\phi(x)\|}, \quad \|\phi(x)\| = \sqrt{\phi^a(x)\phi^a(x)}, \quad a = 1, 2, 3, 4, \tag{17}$$

where $\phi^a(x)$ is a tangent vector field of the axisymmetric black hole spacetime. It is obvious that the singularities of $n^a(x)$ are just the zeros of $\phi^a(x)$. Using

$$\partial_\mu n^a(x) = \frac{1}{\|\phi(x)\|} \partial_\mu \phi^a(x) + \phi^a(x) \partial_\mu \frac{1}{\|\phi(x)\|} \tag{18}$$

and

$$\frac{\partial}{\partial \phi^a} \left( \frac{1}{\|\phi(x)\|} \right) = -\frac{\phi^a}{\|\phi(x)\|^3}, \tag{19}$$

the entropy density $\rho$ introduced in (16) is changed into

$$\rho = -\frac{A}{8} \frac{1}{24\pi^2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\mu \phi^i \partial_\nu \phi^b \partial_\lambda \phi^c \partial_\rho \phi^d \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi^a} \left( \frac{1}{\|\phi(x)\|^2} \right). \tag{20}$$

If we define the Jacobian as

$$\epsilon^{ibcd} J(\phi/x) = \epsilon^{\mu\nu\lambda\rho} \partial_\mu \phi^i \partial_\nu \phi^b \partial_\lambda \phi^c \partial_\rho \phi^d \tag{21}$$

and notice

$$\epsilon_{abcd} \epsilon^{ibcd} = 3! \delta_a^i, \tag{22}$$

we have

$$\rho = -\frac{A}{8}\frac{1}{4\pi^2}\frac{\partial^2}{\partial\phi^a\partial\phi^a}\left(\frac{1}{\|\phi(x)\|^2}\right)J\left(\frac{\phi}{x}\right). \tag{23}$$

Via the general Green's-function relation [15] in $\phi$-space

$$\Delta_\phi\left(\frac{1}{\|\phi(x)\|^2}\right) = -4\pi^2\delta(\phi), \tag{24}$$

where

$$\Delta_\phi = \frac{\partial^2}{\partial\phi^a\partial\phi^a}, \quad a = 1, 2, 3, 4 \tag{25}$$

is the 4-dimensional Laplacian operator in $\phi$-space, we do obtain the $\delta$-function like entropy density $\rho$ of axisymmetric black holes

$$\rho = -\frac{A}{8}\delta(\phi)J\left(\frac{\phi}{x}\right). \tag{26}$$

It is easy to see that $\rho \neq 0$ only when $\phi(x) = 0$. This result shows that the inner structure of entropy of axisymmetric black holes is determined by the zeros of $\phi^a(x)$, i.e. the singularities of $n^a(x)$ which coincide with the timelike Killing vector field of spacetime. So we will expand $\delta(\phi)$ in terms of these zeros in the following.

Suppose that the tangent vector field $\phi^a(x)$ possesses $l$ isolated zeros and let the $i$th zero be $z_i$, i.e.

$$x^\mu = z_i^\mu \quad (\mu = 1, 2, 3, 4,\ i = 1, 2, \ldots, l). \tag{27}$$

Then, as we proved in [16], the $\delta$-function can be expanded by these zeros as

$$\delta(\phi) = \sum_{i=1}^{l}\frac{\beta_i}{|J(\phi/x)_{z_i}|}\delta(x - z_i), \tag{28}$$

where the positive integer $\beta_i$ is called the Hopf index of the $\phi$-mapping at $z_i$ and it means that, when the point $x$ covers the neighborhood of $z_i$ once, the function $\phi(x)$ covers the corresponding region $\beta_i$ times, which is a topological number of the first Chern class and related to the generalized winding number of the $\phi$-mapping. Substituting (28) into (26), the intrinsic topological structure of the entropy density $\rho$ of axisymmetric black holes is formulated by

$$\rho = \frac{A}{8}\sum_{i=1}^{l}\beta_i\eta_i\delta(x - z_i), \tag{29}$$

where

$$\eta_i = \operatorname{sign} J\left(\frac{\phi}{x}\right)\Big|_{z_i} = \pm 1 \tag{30}$$

is called the Brouwer degree of the $\phi$-mapping at $z_i$ [16]. So, from (15), the entropy $S$ of axisymmetric black holes is given by the sum of these Hopf indices and Brouwer degrees

$$S = \frac{A}{8}\sum_{i=1}^{l}\beta_i\eta_i, \tag{31}$$

which is the direct result of the combination of (2) and the Hopf index theorem. In the following, we will calculate the Euler characteristic, the entropy and the latent heat of Kerr–Newman black hole as an example of axisymmetric black holes.

## 4 The Euler Characteristic, the Entropy and the Latent Heat of Kerr–Newman Black Hole

In terms of Boyer–Lindquist coordinates, the Kerr–Newman black hole has the Euclidean metrics

$$ds^2 = \frac{\Delta}{\rho^2}(cdt - a\sin^2\theta d\varphi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\varphi - acdt]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \quad (32)$$

where

$$\Delta = r^2 - 2mr + a^2 + Q^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \quad (33)$$

in which $m, a, Q$ are the mass, the angular momentum per unit mass and the electric charge, respectively. The non-extreme Kerr–Newman black hole has the event horizon $r_+$ and the Cauchy horizon $r_-$ located at

$$r_\pm = m \pm \sqrt{m^2 - a^2 - Q^2}. \quad (34)$$

The extreme case corresponds to

$$r_+ = r_- \quad \text{or} \quad m = \sqrt{a^2 + Q^2}. \quad (35)$$

The vielbein 1-forms of the Kerr–Newman black hole are

$$e^0 = \frac{\sqrt{\Delta}}{\rho}(cdt - a\sin^2\theta d\varphi), \qquad e^1 = \frac{\rho}{\sqrt{\Delta}}dr,$$
$$e^2 = \rho d\theta, \qquad e^3 = \frac{\sin\theta}{\rho}[(r^2+a^2)d\varphi - acdt]. \quad (36)$$

Since the Kerr–Newman black hole has the static axisymmetric properties, we suppose the Killing vector field maintains the same properties of spacetime, which imply

$$\partial_0\phi^\mu = \partial_3\phi^\mu = 0, \quad \mu = 0,1,2,3, \quad (37)$$

using the Killing equation, the (37) and the null vector property of the Killing vector field at the event horizon

$$\phi^a\phi^a|_{r=r_+} = e^a_\mu e^a_\nu \phi^\mu\phi^\nu|_{r=r_+} = g_{\mu\nu}\phi^\mu\phi^\nu|_{r=r_+} = 0, \quad (38)$$

here the superscript '$a$' is the local orthonormal frame index and it relates to the coordinate index '$\mu$' by the vielbein $e^a_\mu$,

$$\phi^a(x) = e^a_\mu(x)\phi^\mu(x), \quad (39)$$

one can solve the Killing vector field $\phi^\mu(x)$ and, by transformed into the local orthonormal vielbein index '$a$', we obtain

$$\phi^0 = (r_+^2 + a^2\cos^2\theta)\sqrt{\Delta}, \qquad \phi^1 = 0, \qquad \phi^2 = 0, \qquad \phi^3 = (r^2 - r_+^2)\sin\theta. \quad (40)$$

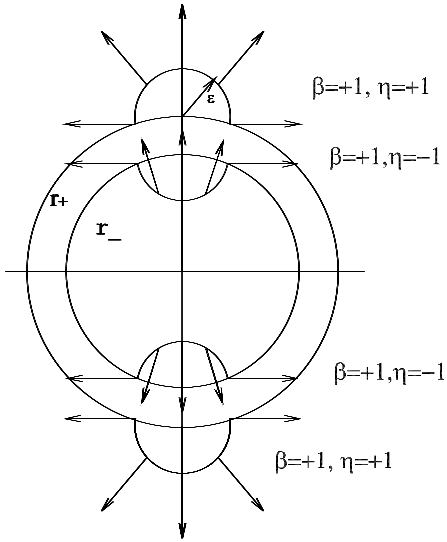


**Fig. 1** The distribution of Killing vector field and the Hopf indices and Brouwer degrees for nonextremal Kerr–Newman black hole

With the aim of finding the Hopf indices and Brouwer degrees of $\phi^a(x)$ $(a = 0, 1, 2, 3)$ at its zeros, one can consider the components $\phi^0(x)$ and $\phi^3(x)$ with variables $r$ and $\theta$ only. It is easy to see that the zeros of non-extreme Kerr–Newman black hole locate at

$$\begin{aligned} &(r = r_+, \theta = 0), \qquad (r = r_-, \theta = 0), \qquad (r = r_+, \theta = \pi), \\ &(r = r_-, \theta = \pi) \quad \text{and} \quad (r = r_+). \end{aligned} \tag{41}$$

We can neglect the case $(r = r_+)$, because the $\delta$-function $\delta(\phi)$ only was expanded by the isolated zeros of $\phi^a(x)$ and we only take into account the contribution of the isolated zeros in (31). What's more, the integral area of the density $\Lambda$ in (4) is taken to be the area outside the horizon, because the observer outside the horizon cannot obtain the information for the division of the area inside the horizon. Therefore the manifold of a black hole is treated as a compact manifold, we do not consider the boundary corrections of the Euler characteristic on the horizon. The distribution of Killing vector field at the neighborhoods of these zeros is shown in Fig. 1, in which $\varepsilon$ denotes the radius of the neighborhoods. For the zero $(r_+, 0)$, under the limit $\varepsilon \to 0$, the Killing vector field $\phi^a(x)$ $(a = 0, 3)$ rotates from 0 to $\pi$ anti-clockwise when the spacetime point $(r, \theta)$ circles the zero in the same way. So we have the Hopf index and the Brouwer degree

$$\beta_1 = 1, \qquad \eta_1 = +1. \tag{42}$$

While for the zero $(r_-, 0)$, when $(r, \theta)$ circles the zero from $\pi$ to 0 anti-clockwise, $\phi^a(x)$ $(a = 0, 3)$ rotates from $\pi$ to 0 clockwise, which leads to the Hopf index and the Brouwer degree

$$\beta_2 = 1, \qquad \eta_2 = -1. \tag{43}$$

Similarly, one can verify the following results:

$$\beta_3 = 1, \qquad \eta_3 = -1, \tag{44}$$

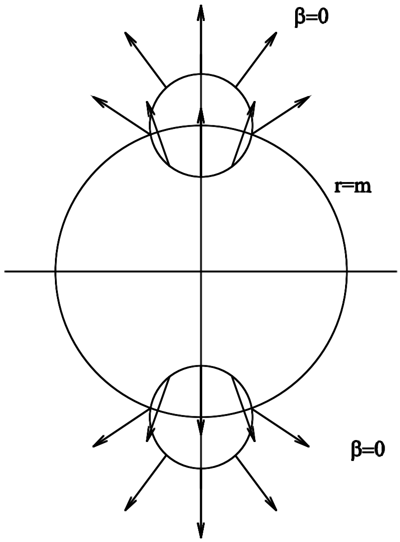


**Fig. 2** The distribution of Killing vector field and the Hopf indices and Brouwer degrees for extremal Kerr–Newman black holes

for the zero $(r_-, \pi)$ and

$$\beta_4 = 1, \qquad \eta_4 = +1, \tag{45}$$

for the zero $(r_+, \pi)$.

Since the event horizon is the boundary of the outer spacetime of Kerr–Newman black hole, it is only the zeros $(r_+, 0)$ and $(r_+, \pi)$ that have contribution to the Euler characteristic and the entropy. Then, in consider the (29) and (31), we obtain the Euler characteristic and the entropy of non-extreme Kerr–Newman black hole

$$\chi = \sum_{i=1,4} \beta_i \eta_i = 2, \tag{46}$$

$$S = \frac{A}{8} \sum_{i=1,4} \beta_i \eta_i = A/4. \tag{47}$$

For the extreme Kerr–Newman black hole, there are only two zeros $(r = m, 0)$ and $(r = m, \pi)$ that contribute to the Euler characteristic and the entropy. The distribution of the Killing vector field is shown in Fig. 2.

In this case, we have the Hopf indices

$$\beta_1 = \beta_2 = 0, \tag{48}$$

and we can obtain the Euler characteristic and the entropy of extreme Kerr–Newman black hole in consider the (29) and (31):

$$\chi = 0, \qquad S = 0. \tag{49}$$

In the following, we will discuss the topological aspect of phase transition of the Kerr–Newman black hole. From (47) and (49) and the Figs. 1, 2, we can see that when the zeros of Killing vector field split or merge with the change of the parameters $m$, $a$ and $Q$, the entropy varies from 0 to $A/4$ discontinuously. This manifests the Kerr–Newman black hole takes place the first-order phase transition, which we call the topological first-order phase transition. Now, we calculate the latent heat of the topological first-order phase transition. From (47) and (49), we get

$$\Delta S = \frac{A}{4}. \tag{50}$$

By the area $A$ of event horizon and the temperature $T$ of Hawking radiation

$$A = 4\pi(r_+^2 + a^2), \tag{51}$$

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\frac{r_+ - r_-}{2(r_+^2 + a^2)}, \tag{52}$$

the latent heat $L$ of the topological first-order phase transition is

$$L = \Delta S \cdot T = \frac{1}{2}\sqrt{m^2 - a^2 - Q^2}. \tag{53}$$

It is obvious that the latent heat is determined by the physical parameters of the Kerr–Newman black hole and it is interesting that the existence condition of $L$ is just that of event horizon in (34). From (53), we can estimate the order of the magnitude of the latent heat of the black hole. For simplicity, we only consider the case that $m$ dominates $L$, which means the latent heat

$$L \doteq \frac{1}{2}m. \tag{54}$$

For the normal range of the mass of the black hole $10^{12}$ kg $\longrightarrow 10^{39}$ kg , the corresponding latent heat are $4.5 \times 10^{35}$ erg $\longrightarrow 4.5 \times 10^{62}$ erg. For example, the mass of the black hole in the central of the galaxy is $6 \times 10^{36}$ kg, so the latent heat is $2.7 \times 10^{60}$ erg. Especially, for the typical black hole evolved from the star whose mass is $10^{31}$ kg, the latent heat is $4.5 \times 10^{54}$ erg, which just lies in the span of the energy of gamma ray bursts $10^{51}$ erg $\longrightarrow 10^{54}$ erg [17–20].

## 5 Conclusions

In this paper, using the relationship between the entropy and the Euler characteristic, we introduce the entropy density to describe the intrinsic topological structure of the entropy of 4-dimensional axisymmetric black holes. From the $\phi$-mapping method, it is shown that the entropy of black holes is determined by the Hopf indices and Brouwer degrees of the singularities of the timelike Killing vector field of spacetime. As an example, the Kerr–Newman black hole is given. For the non-extreme Kerr–Newman black hole, there are only two singularities on the event horizon that have contribution to the Euler characteristic $\chi$ and the entropy $S$ and one obtains $\chi = 2$ and $S = A/4$. For the extreme Kerr–Newman black hole, since the Hopf indices are zero and the singularities have no contribution to the Euler characteristic $\chi$ and the entropy $S$, we obtain $\chi = 0$, $S = 0$. We can see that when the zeros of Killing vector field split or merge with the change of the parameters $m$, $a$ and $Q$, the entropy varies from 0 to $A/4$ discontinuously. This manifests the Kerr–Newman black hole takes place the topological first-order phase transition. The corresponding latent heat is given in (53). In addition, we estimate the order of the magnitude of the latent heat of the black hole. To a typical black hole, the latent heat is $4.5 \times 10^{54}$ erg, which just lies in the span of the energy of gamma ray bursts.

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